Systèmes de Coordonnées

Eric T. Lefèvre*

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*8 rue des Lampes 92190 Meudon
Eric.Lefevre@obspm.fr
etlefevre@chez.com

Contents

Introduction

Ce petit texte donne quelques formules souvent utilisées (gradient, divergence, etc.) pour les trois systèmes de coordonnées les plus utilisés.

1 Coordonnées cartésiennes

Scalaires et vecteurs

$$\begin{aligned} U &= U(x,y,z) \\ \vec{A} &= A_x\vec{e_x} + A_y\vec{e_y} + A_z\vec{e_z} \end{aligned}$$

gradient

$$\vec{\nabla} \equiv \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \quad \vec{A} \equiv \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

$$\overrightarrow{\mathrm{grad}}(U) = \vec{\nabla} U = \frac{\partial U}{\partial x}\vec{e_x} + \frac{\partial U}{\partial y}\vec{e_y} + \frac{\partial U}{\partial z}\vec{e_z}$$

divergence

$$\begin{aligned} \mathrm{div}\,\vec{A} = \vec{\nabla}\cdot\vec{A} &= \frac{\partial}{\partial x}A_x + \frac{\partial}{\partial y}A_y + \frac{\partial}{\partial z}A_z \\ &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \end{aligned}$$

Laplacien scalaire et vectoriel

Laplacien (scalaire)

$$\triangle U = \vec{\nabla}\cdot(\vec{\nabla} U) = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}$$

Laplacien vectoriel

$$\triangle \vec{A} \equiv \begin{pmatrix} \triangle A_x \\ \triangle A_y \\ \triangle A_z \end{pmatrix}$$

Rotationnel

$$\begin{aligned}\overrightarrow{\mathrm{rot}}\,\vec{A} = \vec{\nabla}\wedge\vec{A} &= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\vec{e_x}\\ &+ \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\vec{e_y}\\ &+ \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\vec{e_z}\end{aligned}$$

Eléments de longueur, surface, ...

$$\begin{aligned}\mathrm{d}\ell &= \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2}\\ \mathrm{d}S(x,y) &= \mathrm{d}x\,\mathrm{d}y\\ \mathrm{d}S(x,z) &= \mathrm{d}x\,\mathrm{d}z\\ \mathrm{d}S(y,z) &= \mathrm{d}y\,\mathrm{d}z\\ \mathrm{d}V &= \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\end{aligned}$$

2 Coordonnées cylindriques

Scalaires et vecteurs

$$\begin{aligned}U &= U(\rho,\varphi,z)\\ \vec{A} &= A_\rho\vec{e_\rho} + A_\varphi\vec{e_\varphi} + A_z\vec{e_z}\end{aligned}$$

$$\begin{aligned}A_\rho &= A_x\cos\varphi + A_y\sin\varphi\\ A_\varphi &= -A_x\sin\varphi + A_y\cos\varphi\end{aligned}$$

gradient

$$\vec{\nabla}U = \frac{\partial U}{\partial\rho}\vec{e_\rho} + \frac{1}{\rho}\frac{\partial U}{\partial\varphi}\vec{e_\varphi} + \frac{\partial U}{\partial z}\vec{e_z}$$

divergence

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}$$

Laplacien

$$\triangle U = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial U}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 U}{\partial \varphi^2} + \frac{\partial^2 U}{\partial z^2}$$

Rotationnel

$$\begin{aligned}
\overrightarrow{\mathrm{rot}}\,\vec{A} &= \left(\frac{1}{\rho}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}\right)\vec{e_\rho} \\
&+ \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right)\vec{e_\varphi} \\
&+ \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho A_\varphi) - \frac{\partial A_\rho}{\partial \varphi}\right]\vec{e_z}
\end{aligned}$$

Eléments de longueur, ...

$$\begin{aligned}
\mathrm{d}\ell &= \sqrt{\mathrm{d}\rho^2 + \rho^2 \mathrm{d}\varphi^2 + \mathrm{d}z^2} \\
\mathrm{d}S(\rho, z) &= \mathrm{d}\rho\,\mathrm{d}z \\
\mathrm{d}S(\rho, \varphi) &= \rho\,\mathrm{d}\rho\,\mathrm{d}\varphi \\
\mathrm{d}S(\varphi, z) &= \rho\,\mathrm{d}\varphi\,\mathrm{d}z \\
\mathrm{d}V &= \rho\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}z
\end{aligned}$$

3 Coordonnées sphériques

Scalaires et vecteurs

$$\begin{aligned}
U &= U(r, \theta, \varphi) \\
\vec{A} &= A_r\vec{e_r} + A_\theta\vec{e_\theta} + A_\varphi\vec{e_\varphi}
\end{aligned}$$

$$\begin{aligned}
A_r &= A_\rho \sin\theta + A_z \cos\theta \\
A_\theta &= A_\rho \cos\theta - A_z \sin\theta \\
A_\varphi &= -A_x \sin\varphi + A_y \cos\varphi
\end{aligned}$$

gradient

$$\vec{\nabla} U = \frac{\partial U}{\partial r}\vec{e_r} + \frac{1}{r}\frac{\partial U}{\partial \theta}\vec{e_\theta} + \frac{1}{r\sin\theta}\frac{\partial U}{\partial \varphi}\vec{e_\varphi}$$

divergence

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$$

Laplacien

$$\triangle U = \frac{1}{r}\frac{\partial^2}{\partial r^2}(rU) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial U}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 U}{\partial\varphi^2}$$

Rotationnel

$$\begin{aligned}
\overrightarrow{\mathrm{rot}}\,\vec{A} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta A_\varphi\right) - \frac{\partial A_\theta}{\partial\varphi}\right] \\
&+ \frac{1}{r\sin\theta}\left[\frac{\partial A_r}{\partial\varphi} - \sin\theta\,\frac{\partial}{\partial r}\left(rA_\varphi\right)\right] \\
&+ \frac{1}{r}\left[\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{\partial A_r}{\partial\theta}\right]
\end{aligned}$$

Eléments de longueur, surface, ...

$$\begin{aligned}
\mathrm{d}\ell &= \sqrt{\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\varphi^2} \\
\mathrm{d}S(r,\theta) &= r\mathrm{d}r\mathrm{d}\theta \\
\mathrm{d}S(r,\varphi) &= r\sin\theta\,\mathrm{d}r\,\mathrm{d}\varphi \\
\mathrm{d}S(\theta,\varphi) &= r^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi \\
\mathrm{d}V &= r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi
\end{aligned}$$

4 Formules et propriétés utiles

$$
\begin{aligned}
\overrightarrow{\mathrm{rot}}(\vec{\nabla} U) &= \vec{\nabla} \wedge (\vec{\nabla} U) = \vec{0} \\
\mathrm{div}(\vec{\nabla} U) &= \vec{\nabla} \cdot (\vec{\nabla} U) = \triangle U \\
\mathrm{div}(\overrightarrow{\mathrm{rot}}\,\vec{A}) &= \vec{\nabla} \cdot (\vec{\nabla} \wedge \vec{A}) = 0 \\
\overrightarrow{\mathrm{rot}}(\overrightarrow{\mathrm{rot}}\,\vec{A}) &= \vec{\nabla} \wedge (\vec{\nabla} \wedge \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \triangle \vec{A}\,^{1}
\end{aligned}
$$

$$
\begin{aligned}
\vec{L} &= \frac{\hbar}{i}\,\vec{r} \wedge \vec{\nabla} \qquad \text{moment cinétique} \\
\vec{\nabla} &= \frac{\vec{r}}{r}\,\frac{\partial}{\partial r} - \frac{i}{\hbar r^2}\,\vec{r} \wedge \vec{L} \\
\triangle &= \frac{1}{r}\,\frac{\partial^2}{\partial r^2}\,r - \frac{\vec{L}^2}{\hbar^2 r^2}
\end{aligned}
$$

$$
\begin{aligned}
\vec{C} \cdot (\vec{A} \wedge \vec{B}) &= \vec{A} \cdot (\vec{B} \wedge \vec{C}) = \vec{B} \cdot (\vec{C} \wedge \vec{A}) \\
\vec{A} \wedge (\vec{B} \wedge \vec{C}) &= (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C} \\
\vec{A} \wedge (\vec{B} \wedge \vec{C}) + \vec{B} \wedge (\vec{C} \wedge \vec{A}) &+ \vec{C} \wedge (\vec{A} \wedge \vec{B}) = \vec{0} \\
(\vec{A} \wedge \vec{B}) \cdot (\vec{C} \wedge \vec{D}) &= (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C}) \\
(\vec{A} \wedge \vec{B}) \wedge (\vec{C} \wedge \vec{D}) &= \left[(\vec{A} \wedge \vec{B}) \cdot \vec{D}\right] \vec{C} - \left[(\vec{A} \wedge \vec{B}) \cdot \vec{C}\right] \vec{D} \\
&= \left[(\vec{C} \wedge \vec{D}) \cdot \vec{A}\right] \vec{B} - \left[(\vec{C} \wedge \vec{D}) \cdot \vec{B}\right] \vec{A}
\end{aligned}
$$

$$
\begin{aligned}
\vec{\nabla}(UV) &= U\vec{\nabla} V + V\vec{\nabla} U \\
\triangle(UV) &= U \triangle V + 2\,(\vec{\nabla} U \cdot \vec{\nabla} V) + V \triangle U
\end{aligned}
$$

$$
\begin{aligned}
\vec{\nabla} \cdot (U\vec{A}) &= U\vec{\nabla} A + \vec{A} \cdot \vec{\nabla} U \\
\mathrm{div}(U\vec{A}) &= U\,\mathrm{div}\,\vec{A} + \vec{A} \cdot \overrightarrow{\mathrm{grad}}(U)
\end{aligned}
$$

$$
\begin{aligned}
\vec{\nabla} \wedge (U\vec{A}) &= U\vec{\nabla} \wedge \vec{A} + \vec{\nabla} U \wedge \vec{A} \\
\overrightarrow{\mathrm{rot}}(U\vec{A}) &= U\,\overrightarrow{\mathrm{rot}}\,\vec{A} + \overrightarrow{\mathrm{grad}}\,U \wedge \vec{A}
\end{aligned}
$$

[1]Rappel: il s'agit du Laplacien vectoriel

$$\begin{aligned}
\vec{\nabla}\cdot(\vec{A}\wedge\vec{B}) &= \vec{B}\cdot(\vec{\nabla}\wedge\vec{A}) - \vec{A}\cdot(\vec{\nabla}\wedge\vec{B}) \quad {}^{2} \\
\mathrm{div}(\vec{A}\wedge\vec{B}) &= \vec{B}\cdot\overrightarrow{\mathrm{rot}}\,\vec{A} - \vec{A}\cdot\overrightarrow{\mathrm{rot}}\,\vec{B}
\end{aligned}$$

$$\begin{aligned}
\vec{\nabla}(\vec{A}\cdot\vec{B}) &= \vec{A}\wedge(\vec{\nabla}\wedge\vec{B}) + \vec{B}\wedge(\vec{\nabla}\wedge\vec{A}) + (\vec{B}\cdot\vec{\nabla})\vec{A} + (\vec{A}\cdot\vec{\nabla})\vec{B} \\
\overrightarrow{\mathrm{grad}}(\vec{A}\cdot\vec{B}) &= \vec{A}\wedge\overrightarrow{\mathrm{rot}}\,\vec{B} + \vec{B}\wedge\overrightarrow{\mathrm{rot}}\,\vec{A} + (\vec{B}\cdot\vec{\nabla})\vec{A} + (\vec{A}\cdot\vec{\nabla})\vec{B}
\end{aligned}$$

$$\begin{aligned}
\vec{\nabla}\wedge(\vec{A}\wedge\vec{B}) &= \vec{A}\,(\vec{\nabla}\cdot\vec{B}) - \vec{B}\,(\vec{\nabla}\cdot\vec{A}) + (\vec{B}\cdot\vec{\nabla})\vec{A} + (\vec{A}\cdot\vec{\nabla})\vec{B} \\
\overrightarrow{\mathrm{rot}}(\vec{A}\wedge\vec{B}) &= \vec{A}(\mathrm{div}\,\vec{B}) - \vec{B}(\mathrm{div}\,\vec{A}) + (\vec{B}\cdot\vec{\nabla})\vec{A} + (\vec{A}\cdot\vec{\nabla})\vec{B}
\end{aligned}$$

où $\vec{B}\cdot\vec{\nabla}\,\vec{A}$ est tel que:

$$\left[(\vec{B}\cdot\vec{\nabla})\,\vec{A}\right]_i = B_j\partial_j A_i = \sum_j B_j\frac{\partial A_i}{\partial x_j} \quad (i = x, y, z)$$

$$\begin{aligned}
\vec{\nabla}\left(\frac{1}{U}\right) &= -\frac{\vec{\nabla}U}{U^2} \\
\vec{\nabla}(\ln U) &= \frac{\vec{\nabla}U}{U} \\
\triangle(\ln U) &= \frac{\triangle U}{U} - \frac{(\vec{\nabla}U)^2}{U^2} \\
\triangle(\vec{\nabla}U) &= \vec{\nabla}(\triangle U) \\
\vec{\nabla}\cdot(f\nabla U) &= f\,\triangle\,U + \vec{\nabla}f\cdot\vec{\nabla}U
\end{aligned}$$

[2]Notez la différence avec la formule équivalente pour un vecteur. Ceci montre bien qu'il ne faut pas oublier que $\vec{\nabla}$ est un opérateur vectoriel et pas un simple vecteur.

Index